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Poisson-Lie T-plurality

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Abstract: We extend the path-integral formalism for Poisson-Lie T-duality to include the case of Drinfeld doubles which can be decomposed into bi-algebras in more than one way. We give the correct shift of the dilaton, correcting a mistake in the litterature. We then use the fact that the six dimensional Drinfeld doubles have been classified to write down all possible conformal Poisson-Lie T-duals of three dimensional space times and we explicitly work out two duals to the constant dilaton and zero anti-symmetric tensor Bianchi type V space time and show that they satisfy the string equations of motion. This space-time was previously thought to have no duals because of the tracefulness of the structure constants.

Keywords: String Duality, Superstring Vacua.

Contents

1. Introduction

The importance of target space (or T for short) duality in string theory [1, 2] can hardly be overstated. It has played a fundamental role in gaining non-perturbative information about string theory. In its original appearance it can be understood as a symmetry of the string background equations of motion on a background which has abelian isometries [3]. Given such a background there exists a well defined procedure in the path-integral setting for finding the dual background [4]. Carefully regularizing the determinants that appear when performing the path-integral this procedure also leads to a shift in the dilaton [5] which is necessary for the dual background to be consistent with string theory, i.e. for the dual background to satisfy the string beta functions. Later, this procedure was generalized to the case where the isometries of the background are non-abelian [6]. In the non-abelian case there appeared two new features which are not present in the original abelian duality setting. The first is, as was discovered in [7], that for some backgrounds, even including the dilaton shift, the dual is not conformal (it does not satisfy the beta functions). This was later explained [8, 9] in terms of a mixed gravitational/gauge anomaly which appears when the structure constants of the initial isometry are not traceless ($f_{ab}^{a} \neq 0$). The second puzzling feature was that the dual space-time one ends up with following the standard non-abelian T-dualization procedure in general does not have isometries making it impossible to perform the duality once more to get back to what one started with, thus making the name "duality" somewhat of a misnomer. This second feature was understood in [10] (see also [11]) where Poisson-Lie T-duality was introduced. In that paper it was explained how abelian and non-abelian T-duality can be embedded in a larger algebraic framework where the basic object is a so called Drinfeld double (more on this in the next section). They showed that the sigma models one ends up with are dual to each other by showing that there exists canonical transformations between them. In particular, there is no need for a sigma model to have isometries for the duality to work. However, in this approach it is not clear how the dilaton should transform since the dilaton transformation is a quantum effect. A first look at quantum effects in the context of Poisson-Lie T-duality was made in [12, 13]. In [13] a path-integral approach for Poisson-Lie T-duality was introduced (see also [14]). It was found that just as in non-abelian T-duality [8, 9], a group with traceful structure constants introduces an anomaly which spoils the conformal invariance. Also, the way the dilaton should transform in order to preserve conformal invariance was found. Unfortunately, due to the lack of explicit examples of Drinfeld doubles at that time it was not possible to test this prescription on a concrete example. Therefore a mistake in the dilaton transformation formula was not discovered. Further quantum properties were studied and discussed in [15]–[19]. By now Poisson-Lie T-duality is quite well studied and generalizations have been made in several directions. It has been studied in the context of supersymmetry [20]–[27], in the context of open strings and D-branes [28, 29] and in the context of mirror symmetry [30, 17].

Recently all six dimensional Drinfeld doubles have been classified [31, 32]. This provides a large class of examples on which Poisson-Lie T-duality can be tested. Furthermore, in [31] it was explicitly shown how most Drinfeld doubles actually can be decomposed into bi-algebras in more than one way.[1] This makes the duality much richer - one should maybe speak about plurality instead. Quite surprising is that ordinary non-abelian T-duality where the isometry *does* have traceful structure constants and thus will be anomalous can be embedded in Poisson-Lie T-duality in such a way that if the Drinfeld double admits more than one decomposition and in that decomposition at least one of the algebras have trace free structure constants, it is possible to find a dual *and* conformal background (or backgrounds depending on how many such decompositions there are) to the initial background. The purpose of this paper is to generalize the path-integral formulation of Poisson-Lie T-duality given in [13] to include the case of Drinfeld doubles decomposable in more than one way. We give the correct dilaton and background transformations and use the Bianchi V space time of [7] to show the power of Poisson-Lie T-duality. This space-time was used in [7] to show that non-abelian T-duality does not work when the structure constants are traceful. Here we show how Poisson-Lie T-duality gives us two new space-times which we give explicitly and check that they indeed are solutions to the string equations of motion.

The paper is organized as follows: In section 2 we give definitions of and introduction to Drinfeld doubles and Poisson-Lie T-duality. In section 3 we give the general formalism and derive the general formulas for the background and dilaton shift. In section 4 we give a general discussion of all (possibly) *conformal* Poisson-Lie T-duality chains in three dimensions. In section 5 we calculate explicitly the duals of the Bianchi V space time used in [7] and in section 6 we conclude and discuss open questions.

[1] This was anticipated already in [10].

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2. Drinfeld doubles and T-duality

A Drinfeld double D is defined as a connected Lie group such that its Lie algebra D, equipped with a symmetric ad-invariant bilinear form $\langle \bullet, \bullet \rangle$, can be decomposed into a pair of sub-algebras $\mathcal{G}$ and $\tilde{\mathcal{G}}$, maximally isotropic with respect to $\langle \bullet, \bullet \rangle$ and such that the algebra of the Drinfeld double is the direct sum of the two sub-algebras.

The dimensions of the sub-algebras have to be equal and one can choose a basis in each of the sub-algebras $T_a \in \mathcal{G}$, $\tilde{T}^a \in \tilde{\mathcal{G}}$ such that

$$\begin{aligned} \langle T_a, T_b \rangle &= \langle \tilde{T}^a, \tilde{T}^b \rangle = 0\,, \\ \langle T_a, \tilde{T}^b \rangle &= \delta_a^b. \end{aligned} \tag{2.1}$$

Assuming that the two sub-algebras have the form

$$\begin{aligned} [T_a, T_b] &= f_{ab}^c T_c\,, \\ \left[\tilde{T}^a, \tilde{T}^b\right] &= \tilde{f}_c^{ab} \tilde{T}^c\,. \end{aligned} \tag{2.2}$$

one can use the ad-invariance of $\langle \bullet, \bullet \rangle$ to show that the remaining commutation relations must be

$$\left[T_a, \tilde{T}^b\right] = f_{ca}^b \tilde{T}^c + \tilde{f}_a^{bc} T_c\,. \tag{2.3}$$

Furthermore, the Jacobi identities on the Drinfeld double implies the relation between the structure constants

$$f_{ab}^e \tilde{f}_e^{cd} = f_{ae}^c \tilde{f}_b^{ed} + f_{ae}^d \tilde{f}_b^{ce} - f_{be}^c \tilde{f}_a^{ed} - f_{be}^d \tilde{f}_a^{ce}\,. \tag{2.4}$$

In the seminal paper [10] it was shown that one can use this algebraic structure to extend the notion of a background with isometries to backgrounds with so called generalized isometries. Starting with a sigma model action defined on a Lie group

$$\int \left(g^{-1}\partial g\right)^a E_{ab} \left(g^{-1}\bar{\partial} g\right)^b = \int \partial x^i F_{ij} \bar{\partial} x^j, \tag{2.5}$$

where we have used the left invariant frames to write the model in coordinates $F_{ij} = e_i^a E_{ab} e_j^b$, we say that the background F_{ik} has (non-abelian) isometries if $\mathcal{L}_{v_a} F_{ik} = 0$ for some vector field v_a satisfying $[v_a, v_b] = f_{ab}^c v_c$ However, if the background instead satisfies satisfies

$$\mathcal{L}_{v_c} F_{ik} = F_{ij} v_a^j \tilde{f}_c^{ab} v_b^l F_{lk}\,, \tag{2.6}$$

for some constants $\tilde{f}_c^{ab}$, we say that the background has *generalized* isometries. The consistency condition on the lie derivative $[\mathcal{L}_{v_a}, \mathcal{L}_{v_b}] = \mathcal{L}_{[v_a, v_b]}$ then implies the relation (2.4) showing that this construction leads naturally to the Drinfeld double. In [10] it was shown that it is possible to define an equivalent but dual sigma model with a background $\tilde{F}$ satisfying

$$\mathcal{L}_{\tilde{v}^c} \tilde{F}^{ik} = \tilde{F}^{il} v_l^a f_{ab}^c v_n^b \tilde{F}^{mk}, \tag{2.7}$$

where $\tilde{F}$ is related to F at the "origin" of the Drinfeld double by the relation $\tilde{F}(\tilde{x} = 0) = F^{-1}(x = 0)$ where x are the coordinates on the original space-time and $\tilde{x}$ are the coordinates on the dual.

3. General formalism

The path integral formulation of Poisson-Lie T-duality was given in [13]. The parent action taken as the starting point of the duality is a WZW-model[2] defined on the Drinfeld double but with an extra, chiral constraint. In formulas we have

$$\mathcal{Z} = \sqrt{\det G^{(0)}} \int \mathcal{D}l\delta \left[\langle l^{-1}\partial l, \tilde{T}^a\rangle E^{(0)}_{ab} - \langle l^{-1}\partial l, T_b\rangle\right] e^{-I[l]}\,, \tag{3.1}$$

where $E^{(0)}_{ab}$ is a constant (or possibly dependent on the spectator coordinates[3]) matrix, $G^{(0)}$ is the symmetric part of $E^{(0)}$ and

$$I\,[l] = \frac{1}{4\pi}\int \langle l^{-1}\partial l, l^{-1}\bar{\partial} l\rangle + \frac{1}{12\pi}\int \frac{1}{d}\langle l^{-1}dl, \left[l^{-1}dl, l^{-1}dl\right]\rangle. \tag{3.2}$$

Notice that the generators $(T, \tilde{T})$ defines a "canonical" decomposition of the double. To work with arbitrary decompositions we define the transformation matrix

$$\begin{pmatrix} T \\ \tilde{T} \end{pmatrix} = \begin{pmatrix} F & G \\ H & K \end{pmatrix} \begin{pmatrix} U \\ \tilde{U} \end{pmatrix}. \tag{3.3}$$

The group associated with the generators U will be the group over which the final sigma model is defined and the group associated with the generators $\tilde{U}$ will be the auxiliary group which is integrated out. This formalism includes the canonical decomposition by choosing the decomposition matrix as

$$\begin{pmatrix} F & G \\ H & K \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{3.4}$$

and the canonical dual by choosing the decomposition matrix as

$$\begin{pmatrix} F & G \\ H & K \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{3.5}$$

Defining the matrices

$$\begin{aligned} \mu^{ab}(g) &= \langle g\tilde{U}^a g^{-1}, \tilde{U}^b\rangle, \\ \nu_a^b(g) &= \langle U_a, g\tilde{U}^b g^{-1}\rangle\,, \\ M &= K^t E^{(0)} - G^t\,, \\ N &= F^t - H^t E^{(0)}\,, \end{aligned} \tag{3.6}$$

where g is a group element in the group generated by the U generators, and decomposing the general group element $l = \tilde{g}g$, with $\tilde{g}$ in the group generated by the $\tilde{U}$ generators, the following happens:

[2]Notice that this does not mean that the final sigma models are of WZW type. They are simply sigma models defined on a Lie group.

[3]Spectator coordinates are coordinates that do not explicitly take part in the dualization. For instance, if we discuss 6 dimensional Drinfeld doubles, the duality will be between 3 dimensional spaces which means that for the bosonic string there would be 23 spectator coordinates.

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- The measure $\mathcal{D}l$ being the left invariant Haar measure on the Drinfeld double, splits into

$$\mathcal{D}l = \mathcal{D}\tilde{g}\mathcal{D}g \det \nu^{-1}(g)\,. \tag{3.7}$$

- The delta function can be written as

$$\delta\left[JM - \tilde{A}\left(\nu^{-1}\right)^t (N + \mu\nu M)\right], \tag{3.8}$$

where $J^a = \langle \tilde{U}^a, g^{-1}\partial g\rangle$ and $\tilde{A}_a = \langle U_a, \tilde{g}^{-1}\partial\tilde{g}\rangle$.

- The action, using the Polyakov-Wiegmann identity [33] and the fact that $I[g] = I[\tilde{g}] = 0$ which follows from the skew symmetry of $\langle\bullet,\bullet\rangle$, becomes

$$I[\tilde{g}g] = \frac{1}{4\pi}\int \tilde{A}\left(\nu^{-1}\right)^t \bar{J}\,.$$

The integral over the variable $\tilde{g}$ can be performed by changing variables to $\tilde{A} = \tilde{g}^{-1}\partial\tilde{g}$ but doing so we pick up a non-trivial determinant

$$\mathcal{D}\tilde{g} = \mathcal{D}\tilde{A}\frac{1}{\det\left(\partial + \left[\tilde{A},\cdot\right]\right)}, \tag{3.9}$$

which, if the structure constants of the algebra generated by $\tilde{U}$ has non-zero trace (i.e. $\tilde{f}^b_{ab} \neq 0$), has gauge and gravitational anomalies ruining the T-duality as was observed in [8, 9]. In the following we will assume that the auxiliary group is always anomaly free so that we do not have to worry about these factors.

Assuming that we have chosen the $\tilde{U}$ algebra to be anomaly free, we can go on and integrate out the auxiliary group giving us the sigma model

$$\int \mathcal{D}g \det\nu^{-1}\left[\det\left(\left(\nu^t\right)^{-1}(N + \mu\nu M)\right)\right]^{-1}_{\text{reg}} e^{-\frac{1}{4\pi}\int J\left(NM^{-1}+\mu\nu\right)^{-1}\bar{J}}, \tag{3.10}$$

The determinant in the parenthesis, coming from the functional integration over $\tilde{A}$ is not well defined and needs to be regularized. Following [34, 35] we can write it as

$$\begin{aligned}\left[\det\left(\left(\nu^t\right)^{-1}(N + \mu\nu M)\right)\right]^{-1}_{\text{reg}} &= \det\left(\left(\nu^t\right)^{-1}(N + \mu\nu M)\right)^{-1} \\ &\quad e^{\frac{1}{8\pi}\int R^{(2)}\ln\left(\det\left(\left(\nu^t\right)^{-1}(N+\mu\nu M)\right)\right)}.\end{aligned} \tag{3.11}$$

The determinant in front of the exponential is formally the same as the determinant that we started with but it is now regularized and together with the $\sqrt{\det G^{(0)}}$ and the $\det\nu^{-1}$ in the original measure it gives us a final path-integral measure

$$\mathcal{D}g\frac{\sqrt{\det G^{(0)}}\det E}{\det M}\,. \tag{3.12}$$

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It should be possible to show that this can always be written as $\mathcal{D}g\sqrt{\det G}$ where G is the symmetric part of E so that the measure is always the correct coordinate invariant sigma model measure. We have not done this but instead we have checked it explicitly in every example we have calculated. The factor in the exponential can be absorbed in a shift of the dilaton. In summary this gives us a sigma model with the correct integration measure and the background, including the dilaton shift

$$\begin{aligned} E &= \left(NM^{-1} + \mu\nu\right)^{-1} , \\ \phi &= \phi^{(0)} + \ln\det M - \ln\det E - \ln\det\nu . \end{aligned} \tag{3.13}$$

One can check that choosing the trivial decomposition matrix (3.4), one gets the background $E = ((E^{(0)})^{-1} + \mu\nu)^{-1}$, and choosing the basic dual decomposition matrix (3.5), one gets the background $E = (E^{(0)} + \mu\nu)^{-1}$.[4] Up to the last term in the dilaton shift this agrees with the standard formulas as given in [10, 13]. This term appears when carefully regulating all determinants and is essential for the dual model to be conformal. That this term was left out in [13] is presumably the source of the discrepancy in the dilaton shift of [16].

4. All conformal three dimensional duals

In [31] all 6-dimensional Drinfeld doubles were found. Particular care was taken to identify different decompositions into bi-algebras. In all cases where the same double can be decomposed in different ways an explicit transformation matrix, relating the generators in the two cases, were given. From the classification in [31] we can read off all possible *conformal* duality chains in three dimensions. Using their notation $(G|\tilde{G})$[5] and taking care to always write the auxiliary group to be integrated over in the second position, we find

- $(5_b|9) \leftrightarrow (5_b.ii|8) \leftrightarrow (5_b.ii|7_0)$.
- $(5_b.i|8) \leftrightarrow (5.iii|6_0)$.
- $(9|1) \leftrightarrow (1|9)$.
- $(8|1) \leftrightarrow (1|8) \leftrightarrow (5.iii|8) \leftrightarrow (5.i|7_0) \leftrightarrow (5_i|6_0) \leftrightarrow (5|2.ii)$.
- $(4|2_b.iii) \leftrightarrow (4_b|7_0) \leftrightarrow (4_{-b}.i|6_0)$.
- $(7_a|1) \leftrightarrow (7_a|2.i) \leftrightarrow (7_a|2.ii)$.
- $(6_a|1) \leftrightarrow (6_a|2)$.
- $(5|1) \leftrightarrow (5|2.i) \leftrightarrow (6_0|1) \leftrightarrow (1|6_0) \leftrightarrow (5.ii|6_0)$.

[4]Notice here that the matrices $\mu(g)$ and $\nu(g)$ differ in these two examples. They are always calculated for a particular decomposition since g is defined to be in the group generated by the U generators. Therefore μ and ν are decomposition dependent.

[5]The notation is such that G and $\tilde{G}$ are numbers referring to its Bianchi type. In the case where there are more parameters they are indicated by various indices. For a complete definition see [31]

- $(4|1) \leftrightarrow (4|2.i) \leftrightarrow (4|2.ii) \leftrightarrow (6_0|2) \leftrightarrow (2|6_0) \leftrightarrow (4.ii|6_0)$.
- $(3|1) \leftrightarrow (3|2)$.
- $(7_0|1) \leftrightarrow (1|7_0)$.
- $(7_0|2.i) \leftrightarrow (2.i|7_0)$.
- $(7_0|2.ii) \leftrightarrow (2.ii|7_0)$.
- $(2|1) \leftrightarrow (1|2)$.
- $(2|2.i) \leftrightarrow (2.i|2)$.
- $(2|2.ii) \leftrightarrow (2.ii|2)$.
- $(1|1)$.

This list should be seen as the three dimensional Poisson-Lie equivalent of the orbits of the abelian $O(d, d, R)$ duality groups modulo the automorphisms of each of the non isomorphic decompositions by itself as was discussed in [10]. That is, the full duality group would include the automorphisms of each of the algebras and the elements which takes one from one decomposition to the other in the way indicated above. These elements are given exactly by the matrices in [31]. It it also possible to have subdualities for Drinfeld doubles which contains lower dimensional sub Drinfeld doubles. A first step in trying to find these duality groups was taken in [36].

5. An explicit example. Duals of Bianchi V

As an explicit example we calculate duals of a Bianchi type V space time with constant dilaton and no antisymmetric tensor field

$$\begin{aligned} ds^2 &= -dt^2 + t^2\left(dx^2 + e^{2x}(dy^2 + dz^2)\right), \\ b &= 0\,, \\ \phi &= 0\,. \end{aligned} \tag{5.1}$$

This is a solution to the string beta function equations since the metric is in fact flat. This is an appropriate metric to choose for illustration since it was for this metric that it was first noticed that non-abelian duals of space times with isometry group with non traceless structure constants are not conformal [7]. Later it was discovered that this was because of the mixed gravitational/gauge anomaly coming from the jacobians in the path-integral measure. By taking care and always choosing the auxiliary group to be anomaly free, we are guaranteed to get only conformal dual models.

(5|1):
we take (5|1) as the canonical decomposition The full algebra is given by

$$\begin{aligned}
[T_1, T_2] &= -T_2 & [T_2, T_3] &= 0 & [T_3, T_1] &= T_3 \\
\left[\tilde{T}^1, \tilde{T}^2\right] &= 0 & \left[\tilde{T}^2, \tilde{T}^3\right] &= 0 & \left[\tilde{T}^3, \tilde{T}^1\right] &= 0 \\
\left[T_1, \tilde{T}^1\right] &= 0 & \left[T_2, \tilde{T}^1\right] &= 0 & \left[T_3, \tilde{T}^1\right] &= 0 \\
\left[T_1, \tilde{T}^2\right] &= \tilde{T}^2 & \left[T_2, \tilde{T}^2\right] &= -\tilde{T}^1 & \left[T_3, \tilde{T}^2\right] &= 0 \\
\left[T_1, \tilde{T}^3\right] &= \tilde{T}^3 & \left[T_2, \tilde{T}^3\right] &= 0 & \left[T_3, \tilde{T}^3\right] &= -\tilde{T}^1
\end{aligned} \tag{5.2}$$

The parametrization of a general group element we choose as

$$l = \tilde{g}g = e^{\tilde{x}_3\tilde{T}^3} e^{\tilde{x}_2\tilde{T}^2} e^{\tilde{x}_1\tilde{T}^1} e^{x_3T_3} e^{x_2T_2} e^{x_1T_1} \,, \tag{5.3}$$

giving

$$g^{-1}dg = dx_1T_1 + e^{x_1}dx_2T_2 + e^{x_1}dx_3T_3 \,. \tag{5.4}$$

Calculating the matrices (3.6) for this decomposition, we get

$$\begin{aligned}
\mu &= 0 \,, \\
\nu &= \begin{pmatrix} 1 & -x_2e^{x_1} & -x_3e^{x_1} \\ 0 & e^{x_1} & 0 \\ 0 & 0 & e^{x_1} \end{pmatrix} , \\
M &= E^{(0)} \,, \\
N &= 1 \,,
\end{aligned} \tag{5.5}$$

which gives us an $E = E^{(0)}$ as expected for the canonical decomposition. For this to give us the background (5.1) we see that we have to choose $E^{(0)} = t^2\mathbf{1}$. Since $\mu = 0$ there is no anti-symmetric tensor. However, since we want the total dilaton to be zero we need to choose

$$\phi^{(0)} = \ln\det E + \ln\det\nu - \ln\det M = \ln\det\nu = 2x_1 \,. \tag{5.6}$$

The measure factor can also be calculated. Since $\det G^{(0)} = t^6$ and in this case $\det E = \det M = t^6$ it comes out as

$$\mathcal{D}g\frac{t^3t^6}{t^6} = \mathcal{D}g\sqrt{\det G} \,, \tag{5.7}$$

where G is the symmetric part of E so that the full measure is indeed the correct coordinate invariant measure of the sigma model.

$(6_0|1)$:
from [31] we find the non-trivial decomposition matrix

$$\begin{pmatrix} T \\ \tilde{T} \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} U \\ \tilde{U} \end{pmatrix} . \tag{5.8}$$

The algebra is

$$\begin{aligned}
&[U_1, U_2] = 0 && [U_2, U_3] = U_1 && [U_3, U_1] = -U_2 \\
&\left[\tilde U^1, \tilde U^2\right] = 0 && \left[\tilde U^2, \tilde U^3\right] = 0 && \left[\tilde U^3, \tilde U^1\right] = 0 \\
&\left[U_1, \tilde U^1\right] = 0 && \left[U_2, \tilde U^1\right] = -\tilde U^3 && \left[U_3, \tilde U^1\right] = \tilde U^2 \\
&\left[U_1, \tilde U^2\right] = -\tilde U^3 && \left[U_2, \tilde U^2\right] = 0 && \left[U_3, \tilde U^2\right] = \tilde U^1 \\
&\left[U_1, \tilde U^3\right] = 0 && \left[U_2, \tilde U^3\right] = 0 && \left[U_3, \tilde U^3\right] = 0
\end{aligned} \tag{5.9}$$

Decomposing an arbitrary group element as

$$l = \tilde g g = e^{\tilde y_1 \tilde U^1} e^{\tilde y_2 \tilde U^2} e^{\tilde y_3 \tilde U^3} e^{y_1 U_1} e^{y_2 U_2} e^{y_3 U_3}, \tag{5.10}$$

which is equivalent to a change of coordinates

$$\begin{aligned}
x_1 &= -y_3\,, \\
x_2 &= \frac{1}{2}\left(\tilde y_1 + \tilde y_2\right), \\
x_3 &= \frac{1}{2}\left(y_2 - y_1\right), \\
\tilde x_1 &= -\tilde y_3 - \frac{1}{2}\left(y_1 + y_2\right)\left(\tilde y_1 + \tilde y_2\right), \\
\tilde x_2 &= y_1 + y_2\,, \\
\tilde x_3 &= \tilde y_2 - \tilde y_1\,,
\end{aligned} \tag{5.11}$$

we find

$$\begin{aligned}
g^{-1} dg = {} & (dy_1 \cosh y_3 + dy_2 \sinh y_3) U_1 + \\
& + (dy_1 \sinh y_3 + dy_2 \cosh y_3) U_2 + dy_3 U_3\,.
\end{aligned} \tag{5.12}$$

Calculating the matrices (3.6) for this decomposition, we get

$$\begin{aligned}
\mu &= 0\,, \\
\nu &= \begin{pmatrix} \cosh y_3 & \sinh y_3 & 0 \\ \sinh y_3 & \cosh y_3 & 0 \\ -(y_2 \cosh y_3 + y_1 \sinh y_3) & -(y_1 \cosh y_3 + y_2 \sinh y_3) & 1 \end{pmatrix}, \\
M &= \begin{pmatrix} 0 & -1 & -\frac{t^2}{2} \\ 0 & -1 & \frac{t^2}{2} \\ -t^2 & 0 & 0 \end{pmatrix}, \\
N &= \begin{pmatrix} 0 & -\frac{t^2}{2} & -1 \\ 0 & -\frac{t^2}{2} & 1 \\ -1 & 0 & 0 \end{pmatrix},
\end{aligned} \tag{5.13}$$

giving us the background

$$\begin{aligned}
E &= \begin{pmatrix} \frac{1}{t^2} + \frac{t^2}{4} & \frac{1}{t^2} - \frac{t^2}{4} & 0 \\ \frac{1}{t^2} - \frac{t^2}{4} & \frac{1}{t^2} + \frac{t^2}{4} & 0 \\ 0 & 0 & t^2 \end{pmatrix}, \\
\phi &= \phi^{(0)} + \ln t^2\,.
\end{aligned} \tag{5.14}$$

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To go to a coordinate basis we have to use the left invariant one forms (5.12) to write the metric as

$$
\begin{aligned}
ds^2 &= -dt^2 + \left(\left(\frac{1}{t^2} + \frac{t^2}{4} \right) \cosh 2y_3 + \left(\frac{1}{t^2} - \frac{t^2}{4} \right) \sinh 2y_3 \right) \left(dy_1^2 + dy_2^2 \right) + \\
&\quad + 2 \left(\left(\frac{1}{t^2} + \frac{t^2}{4} \right) \sinh 2y_3 + \left(\frac{1}{t^2} - \frac{t^2}{4} \right) \cosh 2y_3 \right) dy_1 dy_2 + t^2 dy_3^2 \,.
\end{aligned} \tag{5.15}
$$

Since the matrix E is symmetric there is no antisymmetric tensor and to evaluate the total dilaton contribution we have to evaluate $\phi^{(0)}$ in the new coordinates. From (5.11) we find that $\phi^{(0)} = 2x_1 = -2y_3$ which gives the final result

$$
\phi = \ln t^2 - 2y_3 \,. \tag{5.16}
$$

Since $\det E = t^2$ and $\det M = t^4$ in this case, the measure factor is

$$
\mathcal{D}g \frac{t^3 t^2}{t^4} = \mathcal{D}g \sqrt{\det G}, \tag{5.17}
$$

where again G is the symmetric part of E for this model so that the full measure is again the correct coordinate invariant measure of the sigma model.

$(5|2.i)$:
for the decomposition $(5|2.i)$ we have the decomposition matrix

$$
\begin{pmatrix} T \\ \tilde{T} \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 1 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} U \\ \tilde{U} \end{pmatrix}, \tag{5.18}
$$

and the decomposition of a general group element we choose as

$$
l = \tilde{g} g = e^{\tilde{y}_3 \tilde{U}^3} e^{\tilde{y}_2 \tilde{U}^2} e^{\tilde{y}_1 \tilde{U}^1} e^{y_3 U_3} e^{y_2 U_2} e^{y_1 U_1} \,, \tag{5.19}
$$

which is equivalent to a change of coordinates $x_1 = -y_1$, $x_2 = \ldots$.[6] Since the isometry group and the choice of coordinates is the same as in the first case (Bianchi V) the left invariant form is the same (5.4). The matrices (3.6) turn out to be

$$
\begin{aligned}
\mu &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \sinh y_1 \\ 0 & -\sinh y_1 & 0 \end{pmatrix}, \\
\nu &= \begin{pmatrix} 1 & -y_2 e^{y_1} & -y_3 e^{y_1} \\ 0 & e^{y_1} & 0 \\ 0 & 0 & e^{y_1} \end{pmatrix},
\end{aligned}
$$

[6]We only need the expression for x_1 in terms of the new coordinates in this example.

$$
\begin{aligned}
M &= \begin{pmatrix} -t^2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \\
N &= \begin{pmatrix} -1 & 0 & 0 \\ 0 & -t^2 & -\frac{1}{2} \\ 0 & \frac{1}{2} & -t^2 \end{pmatrix},
\end{aligned} \tag{5.20}
$$

leading to a background

$$
\begin{aligned}
E &= \begin{pmatrix} t^2 & 0 & 0 \\ 0 & \frac{4t^2}{4t^4+e^{4y_1}} & \frac{-2e^{2y_1}}{4t^4+e^{4y_1}} \\ 0 & \frac{2e^{2y_1}}{4t^4+e^{4y_1}} & \frac{4t^2}{4t^4+e^{4y_1}} \end{pmatrix}, \\
\phi &= \phi^{(0)} + \ln\left(4t^4 + e^{4y_1}\right) - 2y_1 \,.
\end{aligned} \tag{5.21}
$$

The symmetric part of the matrix E gives the metric in the coordinate basis

$$
ds^2 = -dt^2 + t^2 dy_1^2 + \frac{4t^2 e^{2y_1}}{4t^4 + e^{4y_1}} \left(dy_2^2 + dy_3^2\right), \tag{5.22}
$$

whereas the antisymmetric part of the matrix E gives the antisymmetric tensor

$$
b = -\frac{e^{2y_1}}{2t^2} e^2 \wedge e^3, \tag{5.23}
$$

where now the e's refer to the vierbeins of the full metric (5.22). Finally we get the dilaton by remembering that $\phi^{(0)} = 2x_1 = -2y_1$ which gives the final result

$$
\phi = \ln\left(4t^4 + e^{4y_1}\right) - 4y_1. \tag{5.24}
$$

Using the relevant expressions the measure factor becomes

$$
\mathcal{D}g \frac{t^3 \frac{4t^2}{4t^4+e^{4y_1}}}{t^2} = \mathcal{D}g\sqrt{\det G}, \tag{5.25}
$$

where again G is the symmetric part of E.

These backgrounds have all been checked to satisfy the string beta function equations

$$
\begin{aligned}
0 &= R_{ab} - \nabla_a \nabla_b \phi - \frac{1}{4} H_{acd} H_b^{cd}, \\
0 &= \nabla^c \phi H_{cab} + \nabla^c H_{cab}, \\
0 &= R - 2\nabla^2 \phi - \nabla_a \phi \nabla^a \phi - \frac{1}{12} H_{abc} H^{abc}.
\end{aligned} \tag{5.26}
$$

$(1|6_0)$ and $(5.ii|6_0)$:
the decompositions $(1|6_0)$ and $(5.ii|6_0)$ are problematic for one and the same reason. It turns out that the coordinate transformation that is required to go from the original $(5|1)$ to any of these models is always of the form $x_1 = f(y, \tilde{y})$ so that the dilaton $\phi^{(0)} = 2x_1$ always depends also on the auxiliary coordinates $\tilde{y}$ which are to be integrated over to get a sigma model depending only on y. This phenomenon is particularly interesting for the

$(1|6_0)$ model since it just corresponds to ordinary non-abelian dual of the background $(6_0|1)$ which we just found. The ordinary non-abelian T-dualization procedure ensures that the dual background is conformal in the case where the original dilaton depends on spectators only. In this case, where the dilaton also depends on the coordinates, it is not clear how the procedure should work. For instance, how should one gauge the isometry? It is an open question if the duality can be made to work also in this case.

6. Discussion

We have extended the path integral formulation of Poisson-Lie T-duality to the case of Drinfeld doubles which have more than one decomposition into bi-algebras. By carefully considering and regulating determinants which appear in the process of integrating out various degrees of freedom we have been able to determine which cases gives us conformal dual sigma models and which not. In particular we were able to find the correct dilaton shift correcting a mistake in [13]. As an explicit example we showed that the following space-times are dual to each other

$(5|1)$

$$\begin{aligned} ds^2 &= -dt^2 + t^2\left(dx^2 + e^{2x}(dy^2+dz^2)\right), \\ b &= 0\,, \\ \phi &= 0\,, \end{aligned} \tag{6.1}$$

$(6_0|1)$

$$\begin{aligned} ds^2 &= -dt^2 + \left((\frac{1}{t^2}+\frac{t^2}{4})\cosh 2z + (\frac{1}{t^2}-\frac{t^2}{4})\sinh 2z \right)\left(dx^2+dy^2\right) + \\ &\quad + 2\left((\frac{1}{t^2}+\frac{t^2}{4})\sinh 2z + (\frac{1}{t^2}-\frac{t^2}{4})\cosh 2z \right) dx dy + t^2 dz^2\,, \\ b &= 0\,, \\ \phi &= \ln t^2 - 2z\,, \end{aligned} \tag{6.2}$$

$(5|2.i)$

$$\begin{aligned} ds^2 &= -dt^2 + t^2 dx^2 + \frac{4t^2 e^{2x}}{4t^4+e^{4x}}\left(dy^2+dz^2\right), \\ b &= -\frac{e^{2x}}{2t^2} e^2 \wedge e^3\,, \\ \phi &= \ln\left(4t^4 + e^{4x}\right) - 4x\,. \end{aligned} \tag{6.3}$$

It is interesting to notice that all three spacetimes have zero scalar curvature but only the first one is Ricci flat (it is even completely flat).

We also showed that the possible duals $(1|6_0)$ and $(5.ii|6_0)$ were not possible to get in a closed form since the dilaton $\phi^{(0)}$ depends on one of the coordinates which needs to be integrated out in that case. It would be interesting to understand this better and if it is possible to anyway go ahead and perform the integration. Presumably this would introduce some non-trivial dependence on the conformal factor in the action which could have interesting consequences in relation with the mixed anomaly terms.

A clear task is to check all duals suggested in section 4 and check how the duality group acts. The requirement that the duality group preserves the conformal invariance certainly gives some restrictions and it would be interesting to see if it is possible to write down the groups in closed form. It seems that one needs some constraining principle to get groups of manageable size in general [36].

Global issues were not addressed at all in this paper. This is a known problem also in usual non-abelian duality [8] and we do not expect it to be any easier to solve here. Only in the abelian case do we have full control over these effects and can talk about different descriptions of the same theory. Non-abelian dualities should perhaps rather be viewed as solution generating techniques, producing one string background from another.

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References

[1] K. Kikkawa and M. Yamasaki, *Casimir effects in superstring theories*, *Phys. Lett.* **B 149** (1984) 357.

[2] N. Sakai and I. Senda, *Vacuum energies of string compactified on torus*, *Prog. Theor. Phys.* **75** (1986) 692, erratum *ibid.* **77** (1987) 773.

[3] T.H. Buscher, *A symmetry of the string background field equations*, *Phys. Lett.* **B 194** (1987) 59.

[4] M. Roček and E. Verlinde, *Duality, quotients and currents*, *Nucl. Phys.* **B 373** (1992) 630 [hep-th/9110053].

[5] T.H. Buscher, *Path integral derivation of quantum duality in nonlinear sigma models*, *Phys. Lett.* **B 201** (1988) 466.

[6] X.C. de la Ossa and F. Quevedo, *Duality symmetries from nonabelian isometries in string theory*, *Nucl. Phys.* **B 403** (1993) 377 [hep-th/9210021].

[7] M. Gasperini, R. Ricci and G. Veneziano, *A problem with nonabelian duality?*, *Phys. Lett.* **B 319** (1993) 438 [hep-th/9308112].

[8] A. Giveon and M. Roček, *On nonabelian duality*, *Nucl. Phys.* **B 421** (1994) 173 [hep-th/9308154].

[9] E. Álvarez, L. Álvarez-Gaumé and Y. Lozano, *On nonabelian duality*, *Nucl. Phys.* **B 424** (1994) 155 [hep-th/9403155].

[10] C. Klimčík and P. Ševera, *Dual nonabelian duality and the Drinfeld double*, *Phys. Lett.* **B 351** (1995) 455 [hep-th/9502122].

[11] C. Klimčík, *Poisson-Lie T-duality*, *Nucl. Phys.* **46** *(Proc. Suppl.)* (1996) 116 [hep-th/9509095].

[12] A.Y. Alekseev, C. Klimčík and A.A. Tseytlin, *Quantum Poisson-Lie T-duality and WZNW model*, *Nucl. Phys.* **B 458** (1996) 430 [hep-th/9509123].

[13] E. Tyurin and R. von Unge, *Poisson-Lie T-duality: the path-integral derivation*, *Phys. Lett.* **B 382** (1996) 233 [hep-th/9512025].

[14] C. Klimčík and P. Severa, *Poisson-Lie T-duality and loop groups of drinfeld doubles*, *Phys. Lett.* **B 372** (1996) 65 [hep-th/9512040].

[15] K. Sfetsos, *Poisson-Lie T-duality beyond the classical level and the renormalization group*, *Phys. Lett.* **B 432** (1998) 365 [hep-th/9803019].

[16] J. Balog, P. Forgacs, N. Mohammedi, L. Palla and J. Schnittger, *On quantum T-duality in sigma models*, *Nucl. Phys.* **B 535** (1998) 461 [hep-th/9806068].

[17] S.E. Parkhomenko, *On the quantum Poisson-Lie T-duality and mirror symmetry*, *J. Exp. Theor. Phys.* **89** (1999) 5 [hep-th/9812048].

[18] F. Assaoui and T. Lhallabi, *Supersymmetric quantum corrections and Poisson-Lie T-duality*, *Class. and Quant. Grav.* **18** (2001) 277 [hep-ph/0007337].

[19] A. Bossard and N. Mohammedi, *Poisson-Lie duality in the string effective action*, *Nucl. Phys.* **B 619** (2001) 128 [hep-th/0106211].

[20] K. Sfetsos, *Poisson-Lie T-duality and supersymmetry*, *Nucl. Phys.* **56B** *(Proc. Suppl.)* (1997) 302 [hep-th/9611199].

[21] S.E. Parkhomenko, *Poisson-Lie T-duality in $N = 2$ superconformal field theories*, *Sov. Phys. JETP Lett.* **64** (1996) 877 [hep-th/9612034].

[22] S.E. Parkhomenko, *Poisson-Lie T-duality and $N = 2$ superconformal WZNW models on compact groups*, *Mod. Phys. Lett.* **A 12** (1997) 3091 [hep-th/9705233].

[23] S.E. Parkhomenko, *Poisson-Lie T-duality and complex geometry in $N = 2$ superconformal WZNW models*, *Nucl. Phys.* **B 510** (1998) 623 [hep-th/9706199].

[24] C. Klimčík, *Poisson-Lie T-duality and $(1,1)$ supersymmetry*, *Phys. Lett.* **B 414** (1997) 85 [hep-th/9707194].

[25] C. Klimčík and S. Parkhomenko, *Supersymmetric gauged WZNW models as dressing cosets*, *Phys. Lett.* **B 463** (1999) 195 [hep-th/9906163].

[26] F. Assaoui, N. Benhamou and T. Lhallabi, *Poisson-Lie T-duality in supersymmetric WZNW model*, hep-th/0009024.

[27] S.E. Hjelmeland and U. Lindström, *$N = 1$ supersymmetric path-integral Poisson-Lie duality*, *J. High Energy Phys.* **04** (2001) 027 [hep-th/0102089].

[28] C. Klimčík and P. Severa, *Poisson-Lie T-duality: open strings and D-branes*, *Phys. Lett.* **B 376** (1996) 82 [hep-th/9512124].

[29] C. Klimčík and P. Severa, *Open strings and D-branes in WZNW models*, *Nucl. Phys.* **B 488** (1997) 653 [hep-th/9609112].

[30] S.E. Parkhomenko, *Mirror symmetry as a Poisson-Lie T-duality*, *Mod. Phys. Lett.* **A 13** (1998) 1041 [hep-th/9710037].

[31] L. Šnobl and L. Hlavatý, *Classification of 6-dimensional real Drinfeld doubles*, math.QA/0202210.

[32] M.A. Jafarizadeh and A. Rezaei-Aghdam, *Poisson-Lie T-duality and Bianchi type algebras*, *Phys. Lett.* **B 458** (1999) 477 [hep-th/9903152].

[33] A.M. Polyakov and P.B. Wiegmann, *Theory of nonabelian Goldstone bosons in two dimensions*, *Phys. Lett.* **B 131** (1983) 121.

[34] A.A. Tseytlin, *Effective action of gauged WZW model and exact string solutions*, *Nucl. Phys.* **B 399** (1993) 601 [hep-th/9301015].

[35] A.S. Schwarz and A.A. Tseytlin, *Dilaton shift under duality and torsion of elliptic complex*, *Nucl. Phys.* **B 399** (1993) 691 [hep-th/9210015].

[36] L. Snobl, *On modular spaces of semisimple Drinfeld doubles*, hep-th/0204244.

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